What can we learn from scaling?

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I. INTRODUCTION

Any physical system obeys some laws, the acceleration of a body is determined by the its mass and the force acting on the body. The corresponding law, Newton's law, applies to many systems of different size, i.e. to the motion of planets within our stellar system, to human beings, to a tennis ball, and within some limitations, even to the motion of a collection of atoms. Not every law remains valid over a large span of length scales. Nonetheless, one can learn a lot by asking how different quantities *scale*, i.e. how a quality such as the mass, velocity, etc. depends on the size of the object under consideration. As we go form the macroworld, where physical laws are known, down to the nanoworld or even atomic scale, the scaling approach is useful to understand the very new problems we may be facing at the smaller size regime. Here, I would like to illustrate *scaling* using examples primarily from mechanics.

II. DEFINITION OF LENGTH SCALING

Consider a ping-pong ball having diameter d and a tennis ball having diameter D. Assume that the tennis ball is 2-times bigger than the ping-pong ball, so that $D = 2d$. We may now ask, for example, how much bigger the volume of the tennis ball to that of the ping-pong ball is. Sure, it is 8-times bigger, or 2^3. If the tennis ball were k times larger in its linear dimension, the volume would be k^3 larger. We say that volume scales as the third power of its linear dimension. In scaling we compare always similar looking objects, i.e. both the tennis ball and the ping-pong ball are spheres. To retain the similarity all lengths are scaled, either blown up or squeezed down, similarly. In the following we will be using the letter L to denote a characteristic length of the object, for example the diameter or radius of the balls. We then say that volume scales as L^3. We use the notation:

$$\text{volume} = V \sim L^3 \qquad (1)$$

Similarly, we would write for the surface area

$$\text{area} = A \sim L^2 \qquad (2)$$

III. THE HUMAN AS A MACHINE

How does the power of a human scale with its size?

As a reminder power P is energy E per time t. The available energy in our body scales with its volume. However, our body cannot deliver power that scales as L^3, since this would contradict some basic laws from thermodynamics. Any machine operating close to *thermodynamic equilibrium* has an efficiency smaller than one. Consequently, some of the stored energy has to go into heat and cannot be used to perform work. So our body can heat up for some time, but this cannot go on for ever. If we go jogging we first heat up to reach some equilibrium. From now on, the heat generated in our body must leave our body. Since this heat release is via the skin, it scales as the surface. Therefore the (equlibrium) power of a human (animal) scales as:

$$\text{power human} \sim L^2 \qquad (3)$$

I note, that the peak power can scale with the L^3, but only for a relatively short instance.

How often do we have to eat or drink?

The amount we eat scales with the volume, i.e. the size of our stomach. Therefore, the added amount of energy E_{food} scales with the volume too. After a big dinner we have to digest for a period T_{digest}. Since $E_{\text{food}} = P T_{\text{digest}}$, we find for the scaling of T_{digest}:

$$\text{period without supplied food} \sim L \qquad (4)$$

We may also talk about the rate (or frequency) of eating, which is the inverse of the respective time:

$$\text{rate of eating} \sim L^{-1} \qquad (5)$$

We can learn quite a lot from these simple equations. For example, an animal living in the desert has to wait for a long time to find food, hence, it is better big!

If we compare a baby with an adult, the baby necessarily needs to eat more often (has a larger frequency of eating). That is why parents need to get up during

the night to feed their babies. A fly is in search of food all the time.

If we look into the interior of our body, their or millions of *nanomchines* (proteins) working. It is obvious that these nanomachines can only function if food is around all the time. They are imbedded in a soup of 'food', called ATP.

IV.   FORCE AND ACCELERATION

The gravitational force on earth is given by $F_G = mg$, where g is the gravitational acceleration and taken to be a constant. Therefore,

$$\text{weight} \sim L^3 \qquad (6)$$

However, most forces do scale differently, not with the third power of L. The reason is that the size of structures, for example the steel bars in a bridge, the diameter of our legs, are optimized according to the strength of the material. A bone (or a piece of steel) can sustain forces up to a maximum stress. Stress is defined as the force per area. Therefore, structural forces scale (for constant stress) with the area (the cross section):

$$\text{structural force} \sim L^2 \qquad (7)$$
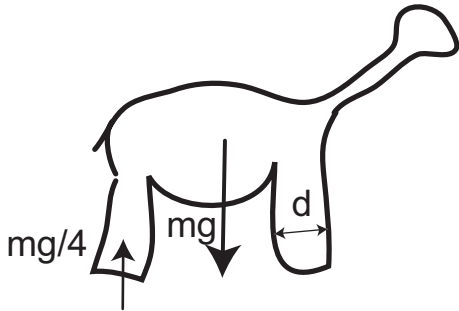
Have a look at the figure. Taking four legs, each one has to carry a force of $mg/4$, where m is the mass of the animal. Since stress is taken to be constant, we have $mg/\pi d^2 = const$. Therefore:

$$\text{leg diameter} \sim L^{3/2}$$

Again, we can learn a lot, both for the large and small scale: big animals have over-proportional legs, small ones have very thin legs.

Going down to the ultimate smallness, i.e. molecular biology, the concept of legs does not make sense anymore. As you know, cells have no legs.

What about acceleration. Galilei found that all objects accelerate exactly identical. Hence, for gravitational forces, the acceleration a is scale independent. This is not true for structural forces. For example, if you ask how fast you can accelerate (or be accelerated in e.g. an accident) without breaking your bones, the answer is:

$$\text{maximum acceleration} \sim L^{-1} \qquad (8)$$

As a consequence, small objects can accelerate fast, large ones cannot. This is the reason why we have difficulties to hit a fly. Because of its small size the fly accelerates much too fast for us to react.

V.   HOW FAST: SPEED AND FREQUENCY

Simple mechanics tells us that the frequency of an oscillator is given by $f = \sqrt{k/m}$, where k is the spring constant. This equation does not yet help us too much, because we do not know the scaling of the spring. If we add n similar springs with spring constant k in series, the effective new spring constant is reduced to k/n. In contrast, if we add the springs in parallel, the new spring constant is enlarged to nk. A piece of flexible material can be considered as a collections of springs. If we think in terms of a rod the number of springs in series scales as the length of the rod. The number of springs in parallel, on the other hand, scales as the cross section, i.e. as L^2. The effective spring constant of a three-dimensional object scales as L^2/L:

$$\text{spring constant} \sim L \qquad (9)$$

It follows from this equation that the frequency scales according to:

$$\text{frequency} \sim L^{-1} \qquad (10)$$

Since the frequency is the inverse of a characteristic time (for a harmonic oscillator the period of motion), the velocity v scales as $v \sim a/f$, where a is the acceleration and f the frequency. Taking what we know from before (equ. 8), we are let to conclude that:

$$\text{velocity} \sim L^0 \qquad (11)$$

(I note here, that this equation is only valid, if there is no friction. We will have a look at friction below.)

It is a good moment to check consistency. Energy is given by work, which is force times distance. If you look at the rate of energy/work per unit time, we get what we call power. Hence, power equals force times velocity: $P = Fv$. As $F \sim L^2$ and $v \sim L^0$, we find $P \sim L^2$ as before.

We learn that the dynamics, determined by the frequency, is fast on the small scale, but net velocities are scale independent provided there is no friction. We know from our own experience, that this scaling is wrong if friction is determining the motion. A feather falls much slower in air than a bottle of wine (if unfortunately it slips out of our hands).

VI.   FRICTION

In Physics-I we learn that the friction force F_R is given by $F_R = \mu F_G$, where μ is the coefficient of friction (a material parameter). This law predicts $F_R \sim L^3$. Though this law is correct, it fails on the small scale (micrometer and nanometer). Friction is due to the adhesion between bodies in contact. It therefore should be determined by the area of contact. The crucial point to realize is that all bodies rest on three legs, even if there are four as in case of a table. Hence, friction is determined by a few points where contact is established. Using continuum mechanics, one can show that the area of contact scales with the body weight. This is the origin of the above law. If we focus on a single contact, or assume that the whole surface is in true contact, which is possible by using a lubricant, the friction force scales with the area of contact:

$$\text{contact friction} \sim L^2 \qquad (12)$$

(This scaling law can also be obtained by saying that shear stress should be scale-independent. Since shear stress is given be the shear force - the frictional force - divided by the area of contact, $F_{\text{shear}} \sim L^2$.)

Consider now a viscous fluid in which the object is embedded. Then, basic physics laws due to Newton tell that the frictional force is determined by the velocity gradient. More explicitly, shear stress is proportional to the velocity gradient. Hence, a constant shear stress, typical velocities scale as:

$$\text{velocity with friction} \sim L^1 \qquad (13)$$

This is indeed different to the scale-independence of velocity without friction.

If the movement of objects is determined by friction, the velocities are getting small in small objects. We know that very small grains of sand, or better dust, can stay in air for a long time. Their velocity relative to the surrounding air is virtually zero. Dust from a volcano eruption can be spread over a whole continent.

VII.   POWER OF A GENERIC MEN-MADE MACHINE

The following figure represents our generic machine. Water streams through a pipe with cross-section A at
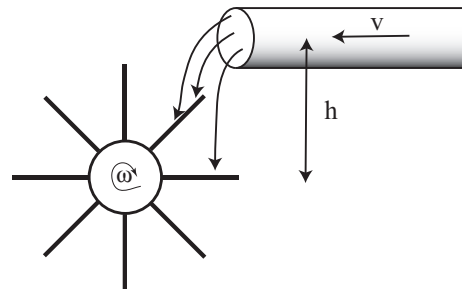
velocity v. For the density of water we use the symbol ρ. Per unit time, the pipe delivers mass, determined by $\rho A v$. This mass falls down, attracted by earth gravitational acceleration g, to keep the rotating wheel in motion. This wheel can deliver mechanical power, which is given by the efficiency of the conversion (a constant) times the gravitational energy, determined by the height h. Hence, we have $P = gh\dot{m} = gh\rho Av \sim L^3$, provided v can be taken as constant. Therefore,

$$\text{machine power, no friction} \sim L^3 \qquad (14)$$

This is equation holds if friction can be neglected. Under this assumption we have learned that the velocity is scale-independent. Usually, this is true for large machines, the ones we human tend to build. It is more convenient to divide the power by the volume of the machine, i.e. to look at the power density:

$$\text{machine power density at no friction} \sim L^0 \qquad (15)$$

Hence, the power density is scale-independent. Note, however, if it comes to the exchange of energy, the situation is different, as I have tried to explain in the first section. Exchange of energy (and information) can at most scale with L^2 (the area).

Let us look at small machines, where friction sets the limit on velocities. Since $v \sim L^1$, we have:

$$\text{machine power density, friction limited} \sim L^1 \qquad (16)$$

The whole molecular bio-machinery is small (actually very small) and functions in a liquid environment in which velocities are limited by friction. Hence, this equation predicts that bio-machines must be inefficient. This appears as a contradiction, because we are taught that biology has had millions of years to perfect itself!?

VIII. POWER OF A MOLECULAR BIO-MACHINE

Molecular motors work different to men-made machines. They operate close to thermal equilibrium. As already mentioned, 'food' in the form of APT is around each bio-machine in excess. The time needed to capture the 'food' is determined by the thermal diffusion time τ_d. If the protein (the molecular machine) is of linear size L, the diffusion time scales as:

$$\text{diffusion time} \sim L^2 \tag{17}$$

The rate of uptake of APT is not only determined by the respective time scale but also by the number of APTs that can bind to the protein. The latter scales with the surface, i.e. with L^2, too. The power is therefore, scale invariant:

$$\text{power density molecular machines} \sim L^{-3} \tag{18}$$

This is the solution of the above problem. Nature has found a way to design machines with very large power densities at a small scale. This scaling-law predicts, that the smaller the better. Indeed, bio-machines are very small proteins.

Very impressive is an absolute comparison. A men-made machine typically delivers a peak power of order $1\,\text{MW/m}^3$. A motor protein can deliver a force of order $1\,\text{nN}$ at a velocity of $10\,\mu\text{m/s}$. This yields a power of $10\,\text{fW}$. Taking for the volume a size of $10\,\text{nm}$, yields for the power density the amazing value of $10\,\text{GW/m}^3$. (I hope, I have not done anything wrong). This number reflects what potentially could be the peak power. This is however (may be also fortunately) never reached. The equilibrium power of a human amounts to approximately $1\,\text{kW/m}^3$ and peak powers (for a very short instant) come close to $1\,\text{MW/m}^3$.

IX. EXERCISE: JUMP HEIGHT

Try to find the scaling-law for the jump height against gravitational acceleration. Do this in an explicit manner using d and l to denote the diameter and length of the legs, and V to denote the volume of the body. You should arrive at the following expression:

$$\text{jump height} \sim \frac{l d^2}{V} \sim L^0 \tag{19}$$

Hence, the jump height is scale invariant. A grasshopper can jump as high as we can do. Sure, such statements have to be taken with caution. A beetle can generally not jump as high.

X. EXERCISE: VELOCITY AGAINST GRAVITATION

The question is: Who can run faster uphill, a small or a large person? Assume that we are asking not for the peak speed, i.e. the one that is possible for a short period, but the steady-state velocity uphill. To answer this question, you have to express the power at a given vertical velocity for a body of mass m against gravitational acceleration g. Since we ask for the steady-state, the power scales as $\sim L^2$. The answer to the question is:

$$\text{running velocity against gravitation} \sim L^{-1} \tag{20}$$

XI. COMPARISON

The following table compares scaling for the mechanical power:

power densities	scaling
equilibrium	L^2
peak, no friction	L^0
peak, (viscous) friction	L^1
peak, molecular machines	L^{-3}

The table illustrates that scaling exponents depend on the underlying physics and can vary quite a lot.

XII. A BIT ELECTRONICS

Scaling laws are particularly important in the field of integrated circuit (IC) technology. As you know, each generation of a memory chip contains more bits. This is only possible, if the size of the electrical elements (transistors, capacitors and resistors) scale down. How they scale determines design rules.

We have started the mechanical part above, by noting that the scaling of structural forces is determined by the strength of the material. The same is true in electronics, in which the electrical field strength has to respect a maximum value beyond which there is what engineers call a breakdown. Hence, the electrical field E is taken to be scale-independent:

$$\text{electric field} \sim L^0 \tag{21}$$

Because voltage V is field times length:

$$\text{voltage} \sim L^1 \tag{22}$$

The mechanical force F acting, for example on a capacitor plate, is given by the field times the area of the plate:

$$\text{force} \sim L^2 \qquad (23)$$

Next, we look at Ohm's law for the resistance $R = \rho l/A$, where ρ is the specific resistivity (a material constant), l the length of the resistor and A its cross-section. Therefore:

$$\text{resistance} \sim L^{-1} \qquad (24)$$

Ohm's law says that the current I is given by V/R:

$$\text{current} \sim L^2 \qquad (25)$$

For the electrical power $P = IV$ we find:

$$\text{power} \sim L^3 \qquad (26)$$

These equations are very nice, because both the current density (current per unit area) and the power density (power per unit volume) are scale invariant, provided the assumption of constant electrical field holds. If these equations would hold circuit engineering of new smaller chips would be simple. However, all scaling laws have limitations. Today, one of the foremost problem in this field of research is the scaling of the resistance. Microfabricated interconnects are very thin, so that the wire thickness can hardly be scaled anymore. Then, $R \sim L^0$ and $I \sim L$. Consequently, the current density will scale as $\sim L^{-1}$ which is disastrous. If the current density increases beyond some threshold the wires are destroyed!

There are many more problems of this kind. In particular quantum mechanical tunneling sets limits on the thickness of gate oxides. Yet another problem is found in the electrical charge, which is used to store the information. Charge scales as L^3. Hence, it rapidly gets very small. If downscaling will go on, a storage capacitor may store only one single electron. At this ultimate limit, scaling must break down.

Without going into details, the future of electronics must consider the quantized nature of matter, the fact that charge is a multiple of single electrons and the fact that localization (to store some charge at one spot in space) is made impossible by quantum-mechanical tunneling.

At the atomic scale, physical laws are very different, sometime surprisingly different. I mention only two recent phenomena:

A guitar string vibrates with some tone. If we increase the tension, the tone gets higher. If we do the same on a string of atoms, i.e. four atoms in a row, the outcome is reversed. If you pull on the string, the vibration frequency is decreased!
Ohm's lay tells that the electrical resistance increases with length. How about single atoms. The resistance of a single atom can be the same as that for two atoms in series. The two atom resistance is not twice the one of a single atom!

XIII. CONCLUDING REMARKS

This lecture serves to demonstrate that scaling-laws can be used to understand the distinct difference in behavior of similar objects at different size. We have used simple physical laws from classical physics. We may use these laws to extrapolate from macrophysics to microphysics. We should however never forget, that all laws have a range of validity. Scaling laws are great to get some insight, but should always be used with caution. In the field of nanoscience, the breakdown of classical scaling is set by quantum mechanics and the granularity of matter (single electron, single atom), so that the laws of large numbers (statistical physics) cannot be used anymore. Actually, the breakdown of classical physics is what makes nanophysics an exciting area of research for physicists. Nanophysics could actually be defined as such.

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